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A NOTE ON TWO RESULTS CONTIGUOUS TO A QUADRATIC TRANSFORMATION DUE TO GAUSS WITH APPLICATIONS

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Abstract. The aim of this paper is to establish two new results contiguous to a well-known, interesting and very useful quadratic transformation due to Gauss. As applications, we first obtain two summation formulas for the series ${}_3F_2$ with unit argument closely related to the classical Watson's summation theorem and then two new identities closely related to that obtained by Krattenthaler and Rao (2003).

Keywords: Gauss hypergeometric function, contiguous function relation, quadratic transformation, Gauss's I and II summation theorems, Watson summation theorem, beta integral method.

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1 Introduction

We start with a well-known, interesting and very useful quadratic transformation due to Gauss (1876), see also Rainville (1971), p. 67, Th. 25, viz.

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix}; z\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}a, \quad \frac{1}{2}b \\ \frac{1}{2}(a+b+1) \end{matrix}; 4z(1-z)\right], \tag{1} $$

provided $|z| < 1$ and $|4z(1-z)| < 1$.

In (1), if we take $z = 1/2$, then the ${}_2F_1$ appearing on the right-hand side can be evaluated with the help of classical Gauss's summation theorem Bailey (1935); Rainville (1971) viz.

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2} $$

provided $\mathrm{Re}(c-a-b) > 0$.

Therefore, we get at once the classical Gauss's second summation theorem Bailey (1935); Rainville (1971) viz.

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1))}. \tag{3} $$

Also, the Beta function $B(\alpha,\beta)$ is defined by the first integral and known to be evaluated as the second one as follows:

$$B(\alpha,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} & (\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0),\\ \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-).\end{cases} \tag{4}$$

In a very interesting, popular and useful research paper Krattenthaler & Rao (2003) made a systematic use of the so-called beta integral method, a method of deriving new hypergeometric identities from old ones by mainly using the beta integral in (4) based on the MATHEMATICA package HYP, to illustrate several interesting identities for the hypergeometric series and Kampé de Fériet series. For this, if we multiply both the sides of (1) by $z^{d-1}(1-z)^{e-d-1}$ and integrate with respect to z over the interval $(0,1)$. After some calculation, evaluating the integral and summing up the series, we get the following interesting identity:

$${}_3F_2\left[\begin{matrix}a, & b, & d\\ \frac{1}{2}(a+b+1), e\end{matrix};1\right]={}_4F_3\left[\begin{matrix}\frac{1}{2}a, & \frac{1}{2}b, & e-d\\ \frac{1}{2}(a+b+1), \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2}\end{matrix};1\right], \tag{5}$$

provided $\mathrm{Re}(d)>0$, $\mathrm{Re}(e-d)>0$ and $\mathrm{Re}(e-d-a/2-b/2+1/2)>0$.

By using the so-called beta integral method, a very special case of the result (5) when a is a nonpositive integer was obtained by Krattenthaler & Rao (2003).

For some recent transformations and generalizations of summation formulas, as well as the corresponding applications to Laplace transforms of convolution type integrals involving Kummer's functions and several special ${}_pF_p$ see Milovanović et al. (2018), Milovanović et al. (2018a) and Milovanović & Rathie (2019).

On the other hand, in the theory of hypergeometric series, contiguous functions relations play an important role. For this, let us denote

$$F={}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};z\right]. \tag{6}$$

Also,

$$F(a+)={}_2F_1\left[\begin{matrix}a+1,b\\ c\end{matrix};z\right] \quad\text{and}\quad F(a-)={}_2F_1\left[\begin{matrix}a-1,b\\ c\end{matrix};z\right], \tag{7}$$

together with similar notations $F(b+)$, $F(b-)$, $F(c+)$ and $F(c-)$.

Gauss (1876) obtained in all fifteen contiguous functions relations between F and any two of its contiguous function (7). They are recorded, for example, in Rainville (1971). One of such relations is given by

$$(a-b)F=aF(a+)-bF(b+). \tag{8}$$

Also, let us define

$$F(a+,b+)={}_2F_1\left[\begin{matrix}a+1,b+1\\ c\end{matrix};z\right] \tag{9}$$

etc.

Then Cho et al. (2001) obtained twenty four contiguous functions relations between F and any one of (7) with any one of (9). In fact, there exists, in all seventy two contiguous functions relations of this type. Therefore, further forty two contiguous functions relations were obtained by Rakha et al. (2011).

In this sequel, it is interesting to mention here that one contiguous function relation between F and any two of (9) is recorded in Rainville (1971). Motivated by this, very recently, Harsh et al. (2018) obtained sixty six contiguous functions relations of such type. Here we shall mention the following one relation that will be required in our present investigation viz.

$$c(a-b)F = a(c-b)F(a+, c+) - b(c-a)F(b+, c+). \tag{10}$$

The aim of this paper is to establish two new results contiguous to (1). As applications, we shall establish the following two summation formulas for the series ${}_3F_2$ with unit argument obtained by a different method by Lavoie et al. (1992)

$$\begin{aligned} {}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{1}{2}(a+b+2), 2c \end{matrix}; 1\right] &= \frac{2^{a+b-1}\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+1)\Gamma(c-\frac{1}{2}a-\frac{1}{2}b)}{(a-b)\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \\ &\times \left\{\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a)\Gamma(c-\frac{1}{2}b+\frac{1}{2})} - \frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}(b+1))}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b)}\right\}, \end{aligned} \tag{11}$$

provided $\mathrm{Re}(2c-a-b) > 0$ and

$$\begin{aligned} {}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{1}{2}(a+b), 2c \end{matrix}; 1\right] &= \frac{2^{a+b-2}\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}(a+b))\Gamma(c-\frac{1}{2}a-\frac{1}{2}b)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \\ &\times \left\{\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a)\Gamma(c-\frac{1}{2}b+\frac{1}{2})} + \frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}(b+1))}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b)}\right\}, \end{aligned} \tag{12}$$

provided $\mathrm{Re}(2c-a-b) > 0$. And two results closely related to (5) are also established.

2 Two new quadratic transformations

In this section, we shall establish two new quadratic transformations closely related to Gauss's quadratic transformation (1) asserted in the following theorem.

Theorem 1. *For $|z| < 1$ and $|4z(1-z)| < 1$, the following results hold true.*

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+2) \end{matrix}; z\right] &= \frac{a}{a-b}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b \\ \frac{1}{2}(a+b+2) \end{matrix}; 4z(1-z)\right] \\ &- \frac{b}{a-b}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}b+\frac{1}{2} \\ \frac{1}{2}(a+b+2) \end{matrix}; 4z(1-z)\right] \end{aligned} \tag{13}$$

and

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b) \end{matrix}; z\right] &= \frac{a}{a+b}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b \\ \frac{1}{2}(a+b+2) \end{matrix}; 4z(1-z)\right] \\ &+ \frac{b}{a+b}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}b+\frac{1}{2} \\ \frac{1}{2}(a+b+2) \end{matrix}; 4z(1-z)\right]. \end{aligned} \tag{14}$$

Proof. In order to establish (13), we proceed as follows. In the contiguous function relation (8), let $c = \frac{1}{2}(a+b+2)$, we have

$${}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+2) \end{matrix}; z\right] = \frac{a}{a-b}\,{}_2F_1\left[\begin{matrix} a+1, \quad b \\ \frac{1}{2}(a+b+2) \end{matrix}; z\right] - \frac{b}{a-b}\,{}_2F_1\left[\begin{matrix} a, \quad b+1 \\ \frac{1}{2}(a+b+2) \end{matrix}; z\right]. \tag{15}$$

Now, if in the two ${}_2F_1$ terms appearing on the right-hand side, we use the result (1), then we get at once the desired result (13).

In exactly the same manner, by taking $c = \frac{1}{2}(a+b)$ in (10) and using (1), we can easily establish the result (14). ☐

Remark 1. *In* (13) *and* (14), *if we take* $z = 1/2$ *and use Gauss's summation theorem* (2), *we get the following summation formulas*

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+2) \end{matrix};\frac{1}{2}\right] \\
&\qquad = \frac{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+1)}{(a-b)}\left\{\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}-\frac{1}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)}\right\}
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b) \end{matrix};\frac{1}{2}\right] \\
&\qquad = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\left\{\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}+\frac{1}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)}\right\},
\end{aligned}
\tag{17}
$$

which were obtained earlier by Lavoie et al. (1996) *by a different method.*

3 Applications

(a) As an application of the newly obtained results (13) and (14), we shall derive the results (11) and (12). For this, we start with the results (Erdelyi et al., 1954, p. 399, Eq. (7))

$$
B(c,d-c)\,{}_3F_2\left[\begin{matrix} a,b,c \\ d,2b \end{matrix};z\right] = \int_0^1 t^{c-1}(1-t)^{d-c-1}\,{}_2F_1\left[\begin{matrix} a,b \\ 2b \end{matrix};zt\right]\,\mathrm{d}t, \tag{18}
$$

valid for $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(d-c) > 0$.

Now, in (18), if we use the result (Rainville, 1971, p. 65, Th. 24)

$$
{}_2F_1\left[\begin{matrix} a,b \\ 2b \end{matrix};2y\right] = (1-y)^{-a}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a,\frac{1}{2}a+\frac{1}{2} \\ b+\frac{1}{2} \end{matrix};\left(\frac{y}{1-y}\right)^2\right]
$$

valid for $|y| < 1/2$ and $|y/(1-y)| < 1$, we have

$$
\begin{aligned}
&B(c,d-c)\,{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ d,\ 2b \end{matrix};z\right] \\
&\quad = \int_0^1 t^{c-1}(1-t)^{d-c-1}\left(1-\frac{zt}{2}\right)^{-a}{}_2F_1\left[\begin{matrix} \frac{1}{2}a,\frac{1}{2}a+\frac{1}{2} \\ b+\frac{1}{2} \end{matrix};\left(\frac{\frac{zt}{2}}{1-\frac{zt}{2}}\right)^2\right]\mathrm{d}t.
\end{aligned}
$$

Now, expressing ${}_2F_1$ as a series, change the order of integration and summation and after some algebra, we have

$$
\begin{aligned}
&B(c,d-c)\,{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ d,\ 2b \end{matrix};z\right] \\
&\quad = \sum_{n=0}^{\infty}\frac{(\frac{1}{2}a)_n\,(\frac{1}{2}a+\frac{1}{2})_n}{(b+\frac{1}{2})_n\,n!}\left(\frac{z}{2}\right)^{2n}\int_0^1 t^{c+2n-1}(1-t)^{d-c-1}\left(1-\frac{zt}{2}\right)^{-a-2n}\mathrm{d}t.
\end{aligned}
$$

Now, if we use the known integral representation for ${}_2F_1$ recorded in Rainville (1971), p. 47, viz.

$$
{}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix};z\right] = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t
$$

valid for $|z| < 1$ and $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, we have, after some simplification

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ d,\ 2b \end{matrix};z\right] = \sum_{n=0}^{\infty} \frac{(c)_{2n}}{(d)_{2n}} \frac{(\frac{1}{2}a)_n\,(\frac{1}{2}a+\frac{1}{2})_n}{(b+\frac{1}{2})_n\,n!} \left(\frac{z}{2}\right)^{2n} {}_2F_1\left[\begin{matrix} a+2n, c+2n \\ d+2n \end{matrix};\frac{z}{2}\right]. $$

Now, interchanging b and c, we have

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ d, 2c \end{matrix};z\right] = \sum_{n=0}^{\infty} \frac{(b)_{2n}}{(d)_{2n}} \frac{(\frac{1}{2}a)_n\,(\frac{1}{2}a+\frac{1}{2})_n}{(c+\frac{1}{2})_n\,n!} \left(\frac{z}{2}\right)^{2n} {}_2F_1\left[\begin{matrix} a+2n, b+2n \\ d+2n \end{matrix};\frac{z}{2}\right]. $$

Further, taking $d = \frac{1}{2}(a+b+2)$, we have

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{1}{2}(a+b+2), 2c \end{matrix};z\right] \\ &\quad = \sum_{n=0}^{\infty} \frac{(b)_{2n}\,(\frac{1}{2}a)_n\,(\frac{1}{2}a+\frac{1}{2})_n}{(\frac{1}{2}(a+b+2))_{2n}\,(c+\frac{1}{2})_n\,n!} \left(\frac{z}{2}\right)^{2n} {}_2F_1\left[\begin{matrix} a+2n,\ b+2n \\ \frac{1}{2}(a+b+4n+2) \end{matrix};\frac{z}{2}\right]. \end{aligned} $$

Using the result (13), we have

$$ \begin{aligned} {}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{1}{2}(a+b+2), 2c \end{matrix};z\right] &= \frac{1}{a-b}\Bigg\{ \sum_{n=0}^{\infty} \frac{(b)_{2n}\,(\frac{1}{2}a)_n\,(\frac{1}{2}a+\frac{1}{2})_n}{(\frac{1}{2}(a+b+2))_{2n}\,(c+\frac{1}{2})_n\,n!} \left(\frac{z}{2}\right)^{2n} \\ &\times \Bigg((a+2n)\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}+n,\ \frac{1}{2}b+n \\ \frac{1}{2}(a+b+4n+2) \end{matrix};2z\left(1-\frac{z}{2}\right)\right] \\ &\quad - (b+2n)\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a+n,\ \frac{1}{2}b+\frac{1}{2}+n \\ \frac{1}{2}(a+b+4n+2) \end{matrix};2z\left(1-\frac{z}{2}\right)\right] \Bigg) \Bigg\}. \end{aligned} $$

Now, setting $z = 1$ and using Gauss's summation theorem (2) and the identity

$$ (a)_{2n} = 2^{2n} \left(\frac{a}{2}\right)_n \left(\frac{a}{2}+\frac{1}{2}\right)_n $$

we have, after some simplification, summing up the series

$$ \begin{aligned} {}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{1}{2}(a+b+2), 2c \end{matrix};1\right] &= \frac{2}{a-b}\,\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right) \\ &- \left(\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2},\ \frac{1}{2}b \\ c+\frac{1}{2} \end{matrix};1\right] \right. \\ &\left. - \frac{1}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b)}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a,\ \frac{1}{2}b+\frac{1}{2} \\ c+\frac{1}{2} \end{matrix};1\right] \right). \end{aligned} $$

Finally, again applying Gauss's summation theorem (2), we arrive at the desired result (11). In exactly the same manner, by employing the result (14), we can derive (12).

Remark 2. *In* (11) *and* (12)*, if we take $c \to \infty$, we recover the results* (16) *and* (17)*, respectively.*

(b) By employing the beta integral method we shall establish the following two new hypergeometric identities:

$$
{}_3F_2\left[\begin{matrix} a, & b, & d \\ \frac{1}{2}(a+b+2), e \end{matrix};1\right] = \frac{a}{a-b}\,{}_4F_3\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}b, & d, & e-d \\ \frac{1}{2}(a+b+2), \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} \end{matrix};1\right]
$$

$$
-\frac{b}{a-b}\,{}_4F_3\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}b+\frac{1}{2}, & d, & e-d \\ \frac{1}{2}(a+b+2), \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} \end{matrix};1\right] \tag{19}
$$

provided $\mathrm{Re}(d) > 0$, $\mathrm{Re}(e-d) > 0$ and $\mathrm{Re}(e-d-a/2-b/2+1) > 0$, and

$$
{}_3F_2\left[\begin{matrix} a, & b, & d \\ \frac{1}{2}(a+b), e \end{matrix};1\right] = \frac{a}{a+b}\,{}_4F_3\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}b, & d, & e-d \\ \frac{1}{2}(a+b+2), \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} \end{matrix};1\right]
$$

$$
+\frac{b}{a+b}\,{}_4F_3\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}b+\frac{1}{2}, & d, & e-d \\ \frac{1}{2}(a+b+2), \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} \end{matrix};1\right], \tag{20}
$$

provided $\mathrm{Re}(d) > 0$, $\mathrm{Re}(e-d) > 0$ and $\mathrm{Re}(e-d-a/2-b/2) > 0$.

Proof. The proofs of (19) and (20) are quite straightforward. For establishing (19), we multiply both sides of (13) by $z^{d-1}(1-z)^{e-d-1}$ and integrating with respect to z over the interval $(0,1)$. After some calculation, evaluating the integral and summing up the series, we get

$$
L.H.S. = \frac{\Gamma(d)\Gamma(e-d)}{\Gamma(e)}\,{}_3F_2\left[\begin{matrix} a, & b, & d \\ \frac{1}{2}(a+b+2), e \end{matrix};1\right] \tag{21}
$$

and

$$
R.H.S. = \frac{\Gamma(d)\Gamma(e-d)}{\Gamma(e)}\left\{\frac{a}{a-b}\,{}_4F_3\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}b, & d, & e-d \\ \frac{1}{2}(a+b+2), \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} \end{matrix};1\right]\right. \tag{22}
$$

$$
\left.-\frac{b}{a-b}\,{}_4F_3\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}b+\frac{1}{2}, & d, & e-d \\ \frac{1}{2}(a+b+2), \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} \end{matrix};1\right]\right\}.
$$

Finally, equating (21) and (22), we arrive at (19).

In the same manner, the result (20) can be proven from the result (14). □

Remark 3. *The results* (19) *and* (20) *are closely related to* (5).

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